# A diagrammatic calculus of syllogisms

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## Categorical Propositions

$\mathbf{A}_{SP}$: All $S$ is $P$ (universal affirmative)

$\mathbf{E}_{SP}$: No $S$ is $P$ (universal negative)

$\mathbf{I}_{SP}$: Some $S$ is $P$ (particular affirmative)

$\mathbf{O}_{SP}$: Some $S$ is not $P$ (particular negative)

## Syllogisms

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- $M$ must appear in both $P_1$ and $P_2$ but is not allowed to appear in $C$.
- $S$ must appear in both $P_2$ and $C$, as the subject of the latter.
- $P$ must appear in both $P_1$ and $C$, as the predicate of the latter.

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**Example:**

$$\mathbf{O}_{PM}, \mathbf{E}_{MS} \vDash \mathbf{I}_{SP}$$

## Valid Syllogisms

| | | | |
|---|---|---|---|
| $\mathbf{A}_{MP}, \mathbf{A}_{SM} \vDash \mathbf{A}_{SP}$<br>$\mathbf{E}_{MP}, \mathbf{A}_{SM} \vDash \mathbf{E}_{SP}$<br>$\mathbf{A}_{MP}, \mathbf{I}_{SM} \vDash \mathbf{I}_{SP}$<br>$\mathbf{E}_{MP}, \mathbf{I}_{SM} \vDash \mathbf{O}_{SP}$ | $\mathbf{E}_{PM}, \mathbf{A}_{SM} \vDash \mathbf{E}_{SP}$<br>$\mathbf{A}_{PM}, \mathbf{E}_{SM} \vDash \mathbf{E}_{SP}$<br>$\mathbf{E}_{PM}, \mathbf{I}_{SM} \vDash \mathbf{O}_{SP}$<br>$\mathbf{A}_{PM}, \mathbf{O}_{SM} \vDash \mathbf{O}_{SP}$ | $\mathbf{I}_{MP}, \mathbf{A}_{MS} \vDash \mathbf{I}_{SP}$<br>$\mathbf{A}_{MP}, \mathbf{I}_{MS} \vDash \mathbf{I}_{SP}$<br>$\mathbf{O}_{MP}, \mathbf{A}_{MS} \vDash \mathbf{O}_{SP}$<br>$\mathbf{E}_{MP}, \mathbf{I}_{MS} \vDash \mathbf{O}_{SP}$ | $\mathbf{A}_{PM}, \mathbf{E}_{MS} \vDash \mathbf{E}_{SP}$<br>$\mathbf{I}_{PM}, \mathbf{A}_{MS} \vDash \mathbf{I}_{SP}$<br>$\mathbf{E}_{PM}, \mathbf{I}_{MS} \vDash \mathbf{O}_{SP}$ |
| $\mathbf{A}_{MP}, \mathbf{A}_{SM} \vDash \mathbf{I}_{SP}$<br>$\mathbf{E}_{MP}, \mathbf{A}_{SM} \vDash \mathbf{O}_{SP}$ | $\mathbf{A}_{PM}, \mathbf{E}_{SM} \vDash \mathbf{O}_{SP}$<br>$\mathbf{E}_{PM}, \mathbf{A}_{SM} \vDash \mathbf{O}_{SP}$ | $\mathbf{A}_{MP}, \mathbf{A}_{MS} \vDash \mathbf{I}_{SP}$<br>$\mathbf{E}_{MP}, \mathbf{A}_{MS} \vDash \mathbf{O}_{SP}$ | $\mathbf{A}_{PM}, \mathbf{E}_{MS} \vDash \mathbf{O}_{SP}$<br>$\mathbf{E}_{PM}, \mathbf{A}_{MS} \vDash \mathbf{O}_{SP}$<br>$\mathbf{A}_{PM}, \mathbf{A}_{MS} \vDash \mathbf{I}_{SP}$ |

# The calculus

$$S \xrightarrow{\mathbf{A}_{SP}} P \qquad\qquad S \longrightarrow \overset{\mathbf{E}_{SP}}{\bullet} \longleftarrow P$$

$$S \longleftarrow \overset{\mathbf{I}_{SP}}{\bullet} \longrightarrow P \qquad\qquad S \longleftarrow \bullet \xrightarrow{\mathbf{O}_{SP}} \bullet \longleftarrow P$$

## Example

$$\mathbf{A}_{MP}, \mathbf{A}_{SM} \vDash \mathbf{A}_{SP}$$

# Example

$$\mathbf{A}_{MP}, \mathbf{A}_{SM} \models \mathbf{A}_{SP}$$

$$S \xrightarrow{\mathbf{A}_{SM}} M \qquad M \xrightarrow{\mathbf{A}_{MP}} P$$

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$$S \xrightarrow{\qquad \mathbf{A}_{SP} \qquad} P$$

# Example

$$\mathbf{E}_{PM}, \mathbf{I}_{SM} \models \mathbf{O}_{SP}$$

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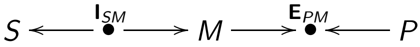

# Example

$\mathbf{E}_{PM}, \mathbf{I}_{SM} \vDash \mathbf{O}_{SP}$

$$S \longleftarrow \overset{\mathbf{I}_{SM}}{\bullet} \longrightarrow M \qquad M \longrightarrow \overset{\mathbf{E}_{PM}}{\bullet} \longleftarrow P$$

$$S \longleftarrow \overset{\mathbf{I}_{SM}}{\bullet} \longrightarrow M \longrightarrow \overset{\mathbf{E}_{PM}}{\bullet} \longleftarrow P$$

$$S \longleftarrow \bullet \xrightarrow{\quad \mathbf{O}_{SP} \quad} \bullet \longleftarrow P$$

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$$S \longleftarrow \bullet \xrightarrow{\mathbf{O}_{SM}} \bullet \longleftarrow M \qquad M \xleftarrow{\mathbf{A}_{PM}} P$$

## Example

$$\mathbf{A}_{PM}, \mathbf{O}_{SM} \vDash \mathbf{O}_{SP}$$

$$S \longleftarrow \bullet \xrightarrow{\mathbf{O}_{SM}} \bullet \longleftarrow M \qquad M \xleftarrow{\mathbf{A}_{PM}} P$$

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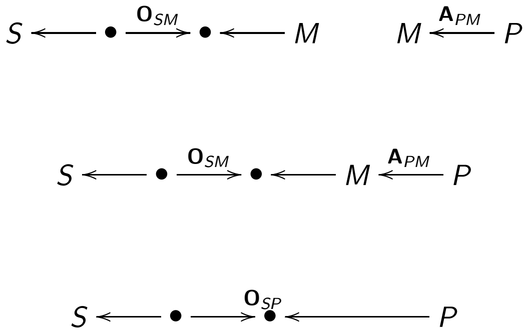
# Example

$\mathbf{A}_{PM}, \mathbf{O}_{SM} \vDash \mathbf{O}_{SP}$



# Example

$$\mathbf{A}_{MP}, \mathbf{A}_{MS} \models \mathbf{I}_{SP}$$

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$$S \xleftarrow{\mathbf{A}_{MS}} M \qquad M \xrightarrow{\mathbf{A}_{MP}} P$$

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$\mathbf{A}_{MP}, \mathbf{A}_{MS} \vDash \mathbf{I}_{SP}$

$$S \xleftarrow{\mathbf{A}_{MS}} M \qquad M \xrightarrow{\mathbf{A}_{MP}} P$$

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$$S \longleftarrow \overset{\mathbf{I}_{SP}}{\bullet} \longrightarrow P$$

## Terminology and notation

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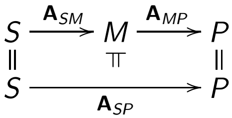
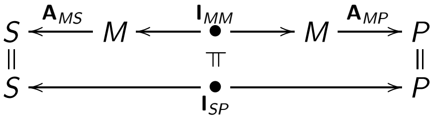
Syllogistic inferences will be henceforth represented by planar diagrams like



## Recovering the valid syllogisms

**Theorem**
A syllogism is valid if and only if there is a (necessarily unique) syllogistic inference from its premisses to its conclusion.

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**Sketch of proof:** A syllogistic inference yields a diagram between $\mathbf{A}_{SP}$, $\mathbf{E}_{SP}$, $\mathbf{I}_{SP}$, $\mathbf{O}_{SP}$ in exactly the following cases:

| | |
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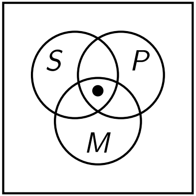
**Example:** $\mathbf{E}_{MP}, \mathbf{I}_{SM} \vDash \mathbf{O}_{SP}$



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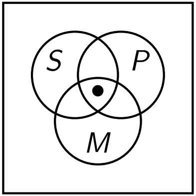


- the calculus easily extends to *n*-term syllogisms.

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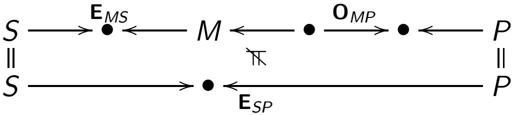

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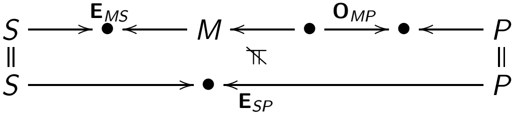


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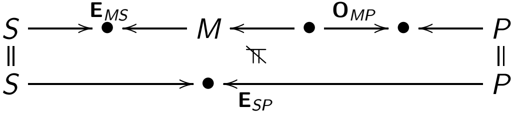


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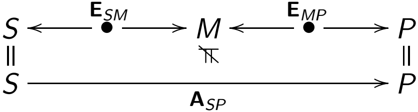

## The rules of syllogism

- From two negative premisses nothing can be inferred.
- From two particular premisses nothing can be inferred.
- From a particular first premise and a negative second premise nothing can be inferred.
- If one premise is particular, then the conclusion is such.
- The conclusion of a syllogism is negative if and only if so is one of its premisses.

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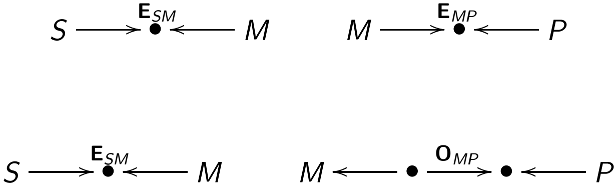

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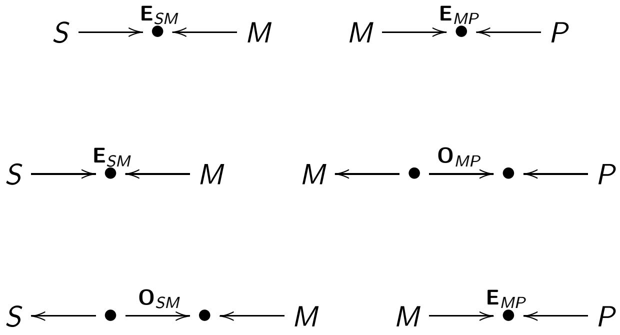

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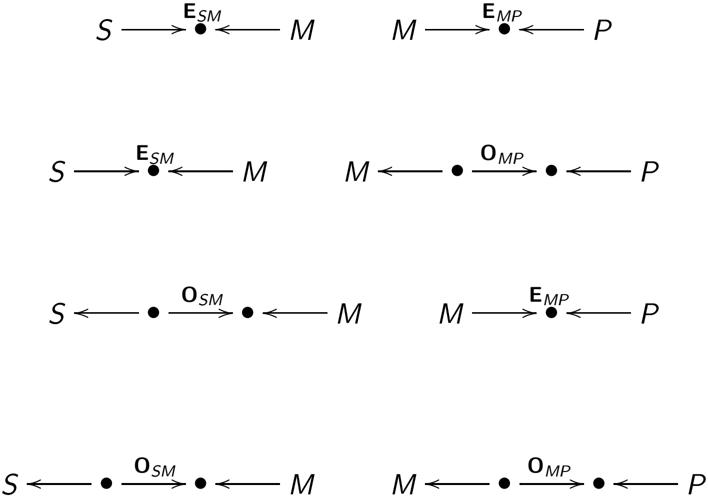
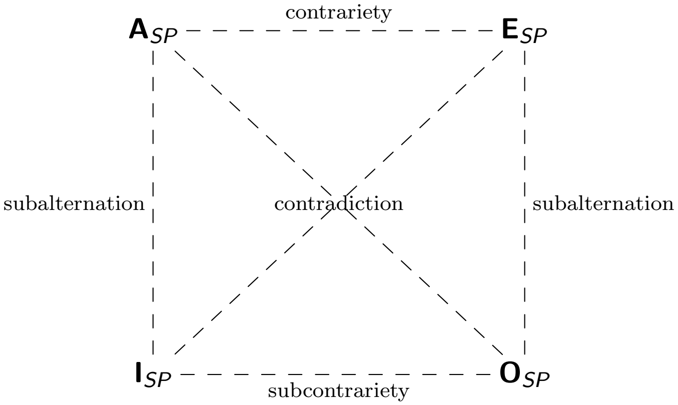


# The Square of Opposition



## The laws of the Square of Opposition

$$\left.\begin{array}{l}\mathbf{A}_{SP}, \mathbf{I}_{SS} \vDash \mathbf{I}_{SP} \\ \\ \mathbf{E}_{SP}, \mathbf{I}_{SS} \vDash \mathbf{O}_{SP}\end{array}\right\} \text{Subalternation}$$

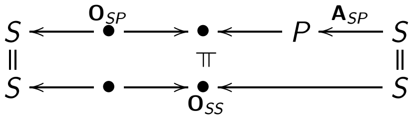
$$\left.\begin{array}{l}\mathbf{A}_{SP}, \mathbf{O}_{SP} \vDash \mathbf{O}_{SS} \\ \\ \mathbf{E}_{SP}, \mathbf{I}_{SP} \vDash \mathbf{O}_{SS}\end{array}\right\} \text{Contradiction}$$

$$\mathbf{E}_{PP}, \mathbf{A}_{SP} \vDash \mathbf{E}_{SP} \ \text{Contrariety}$$

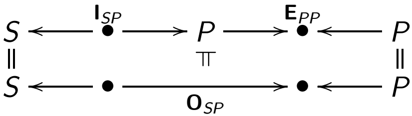
$$\mathbf{E}_{PP}, \mathbf{I}_{SP} \vDash \mathbf{O}_{SP} \ \text{Subcontrariety}$$

## The laws of the Square of Opposition

Contradiction:




Subcontrary:



## Extending the calculus: *n*-term syllogisms

An *n*-term syllogism is a rule of inference

$$P_1, \ldots, P_{n-1} \vDash C$$

in which $P_1, \ldots, P_{n-1}, C$ are categorical propositions any two contiguous of which have exactly one term in common.

**Notation:** the appearing terms will be henceforth denoted by $T_1, T_2, \ldots, T_n$.

## Recovering the valid $n$-term syllogisms

**Theorem**

For every positive natural number $n$, an $n$-term syllogism is valid if and only if there is a (not necessarily unique) pasting of syllogistic inferences from its premisses to its conclusion.

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**Sketch of proof:** For every positive natural number $n$, a syllogistic inference yields a diagram between $\mathbf{A}_{SP}$, $\mathbf{E}_{SP}$, $\mathbf{I}_{SP}$, $\mathbf{O}_{SP}$ in exactly the following cases:

- $T_1 \to T_2 \to \cdots \to T_i \to T_{i+1} \to \cdots \to T_{n-1} \to T_n$.
- $T_1 \to T_2 \to \cdots \to T_i \to \bullet \leftarrow T_{i+1} \leftarrow \cdots \leftarrow T_{n-1} \leftarrow T_n$, with $1 \leq i \leq n-1$.
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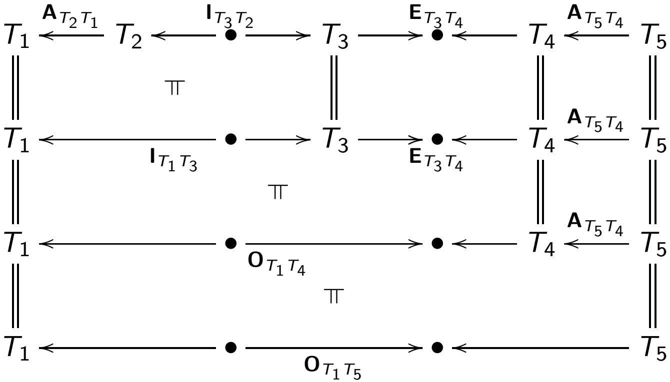
- $T_1 \leftarrow T_2 \leftarrow \cdots \leftarrow T_i \leftarrow \bullet \to \bullet \leftarrow T_{i+1} \leftarrow \cdots \leftarrow T_{n-1} \leftarrow T_n$, with $1 \leq i \leq n-1$.
- $T_1 \leftarrow \cdots \leftarrow T_i \leftarrow \bullet \to T_{i+1} \to \cdots \to T_{j-1} \to \bullet \leftarrow T_j \leftarrow \cdots \leftarrow T_n$, with $1 \leq i < j \leq n$.
- $T_1 \leftarrow \cdots \leftarrow T_i \leftarrow \bullet \to T_i \to \cdots \to T_{j-1} \to \bullet \leftarrow T_j \leftarrow \cdots \leftarrow T_n$, with $1 \leq i < j \leq n$.

# Example

$$\mathbf{A}_{T_5 T_4}, \mathbf{E}_{T_3 T_4}, \mathbf{I}_{T_3 T_2}, \mathbf{A}_{T_2 T_1} \vDash \mathbf{O}_{T_1 T_5}$$



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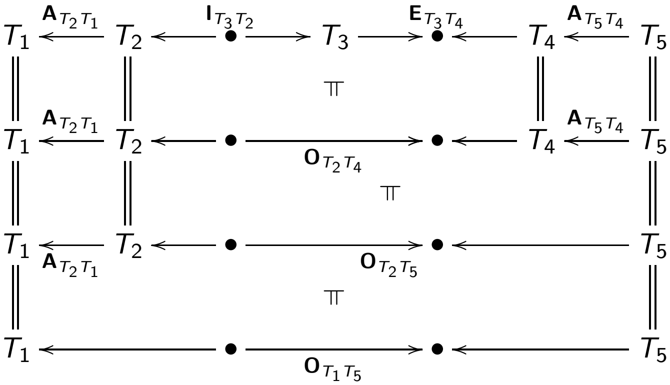

## Valid $n$-term syllogisms

The valid $n$-term syllogisms are $3n^2 - n$:

| syllogism | quantity |
|---|---|
| $\mathbf{A}_{T_{n-1}T_n}, \ldots, \mathbf{A}_{T_1T_2} \vDash \mathbf{A}_{T_1T_n}$ | 1 |
| $\mathbf{A}_{T_nT_{n-1}}, \ldots, \mathbf{E}_{T_iT_{i+1}}, \ldots, \mathbf{A}_{T_1T_2} \vDash \mathbf{E}_{T_1T_n}$ | n-1 |
| $\mathbf{A}_{T_nT_{n-1}}, \ldots, \mathbf{E}_{T_{i+1}T_i}, \ldots, \mathbf{A}_{T_1T_2} \vDash \mathbf{E}_{T_1T_n}$ | n-1 |
| $\mathbf{A}_{T_{n-1}T_n}, \ldots, \mathbf{I}_{T_iT_{i+1}}, \ldots, \mathbf{A}_{T_2T_1} \vDash \mathbf{I}_{T_1T_n}$ | n-1 |
| $\mathbf{A}_{T_{n-1}T_n}, \ldots, \mathbf{I}_{T_{i+1}T_i}, \ldots, \mathbf{A}_{T_2T_1} \vDash \mathbf{I}_{T_1T_n}$ | n-1 |
| $\mathbf{A}_{T_{n-1}T_n}, \ldots, \mathbf{I}_{T_iT_i}, \ldots, \mathbf{A}_{T_2T_1} \vDash \mathbf{I}_{T_1T_n}$ | n |
| $\mathbf{A}_{T_nT_{n-1}}, \ldots, \mathbf{O}_{T_iT_{i+1}}, \ldots, \mathbf{A}_{T_2T_1} \vDash \mathbf{O}_{T_1T_n}$ | n-1 |
| $\mathbf{A}_{T_nT_{n-1}}, \ldots, \mathbf{E}_{T_{j-1}T_j}, \ldots, \mathbf{I}_{T_iT_{i+1}}, \ldots, \mathbf{A}_{T_2T_1} \vDash \mathbf{O}_{T_1T_n}$ | $\frac{(n-1)(n-2)}{2}$ |
| $\mathbf{A}_{T_nT_{n-1}}, \ldots, \mathbf{E}_{T_{j-1}T_j}, \ldots, \mathbf{I}_{T_{i+1}T_i}, \ldots, \mathbf{A}_{T_2T_1} \vDash \mathbf{O}_{T_1T_n}$ | $\frac{(n-1)(n-2)}{2}$ |
| $\mathbf{A}_{T_nT_{n-1}}, \ldots, \mathbf{E}_{T_jT_{j-1}}, \ldots, \mathbf{I}_{T_iT_{i+1}}, \ldots, \mathbf{A}_{T_2T_1} \vDash \mathbf{O}_{T_1T_n}$ | $\frac{(n-1)(n-2)}{2}$ |
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| $\mathbf{A}_{T_nT_{n-1}}, \ldots, \mathbf{E}_{T_{j-1}T_j}, \ldots, \mathbf{I}_{T_iT_i}, \ldots, \mathbf{A}_{T_2T_1} \vDash \mathbf{O}_{T_1T_n}$ | $\frac{n(n-1)}{2}$ |
| $\mathbf{A}_{T_nT_{n-1}}, \ldots, \mathbf{E}_{T_jT_{j-1}}, \ldots, \mathbf{I}_{T_iT_i}, \ldots, \mathbf{A}_{T_2T_1} \vDash \mathbf{O}_{T_1T_n}$ | $\frac{n(n-1)}{2}$ |

**Particular cases:** $n = 1, 2$

$n = 1$: two valid syllogisms, that is the *laws of identity*

$$\mathbf{A}_{T_1T_1} \vDash \mathbf{A}_{T_1T_1} \quad \mathbf{I}_{T_1T_1} \vDash \mathbf{I}_{T_1T_1}$$

$n = 2$: ten valid syllogisms,

- $\mathbf{A}_{T_1T_2} \vDash \mathbf{A}_{T_1T_2}$, $\mathbf{E}_{T_1T_2} \vDash \mathbf{E}_{T_1T_2}$, $\mathbf{I}_{T_1T_2} \vDash \mathbf{I}_{T_1T_2}$, $\mathbf{O}_{T_1T_2} \vDash \mathbf{O}_{T_1T_2}$, plus the *laws of subalternation* $\mathbf{A}_{T_1T_2}, \mathbf{I}_{T_1T_1} \vDash \mathbf{I}_{T_1T_2}$, $\mathbf{E}_{T_1T_2}, \mathbf{I}_{T_1T_1} \vDash \mathbf{O}_{T_1T_2}$.
- $\mathbf{E}_{T_2T_1} \vDash \mathbf{E}_{T_1T_2}$, $\mathbf{I}_{T_2T_1} \vDash \mathbf{I}_{T_1T_2}$ which are the *laws of simple conversion*, and $\mathbf{I}_{T_2T_2}, \mathbf{A}_{T_2T_1} \vDash \mathbf{I}_{T_1T_2}$, $\mathbf{E}_{T_2T_1}, \mathbf{I}_{T_1T_1}, \vDash \mathbf{O}_{T_1T_2}$ which are the *laws of conversion per accidens*.

## $n$-graphs

An *n-graph* is a diagram of sets and functions

$$G_0 \overset{s_0}{\underset{t_0}{\leftleftarrows}} G_1 \overset{s_1}{\underset{t_1}{\leftleftarrows}} G_2 \cdots G_{n-1} \overset{s_{n-1}}{\underset{t_{n-1}}{\leftleftarrows}} G_n$$

such that $s_k \circ s_{k+1} = s_k \circ t_{k+1}$ and $t_k \circ s_{k+1} = t_k \circ t_{k+1}$.

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- A 0-graph is just a set.
- A 1-graph is an ordinary multilabelled oriented graph.

## Categories

A *category* is a multilabelled oriented graph $G_0 \underset{t_0}{\overset{s_0}{\leftleftarrows}} G_1$ equipped with a composition operation on arcs, required to be associative and with neutral element.

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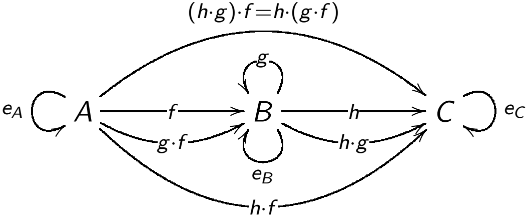
**Example:**



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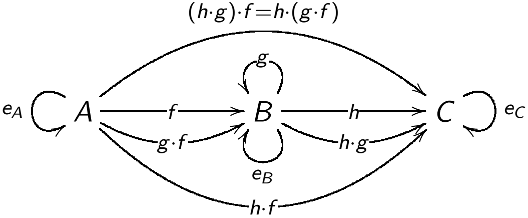


**Examples:** sets and functions, each monoid, well formed formulas and logical consequence. . .

## The free category over a graph

Let $G = (G_0 \underset{t_0}{\overset{s_0}{\leftleftarrows}} G_1)$ be a graph. A *word* in $G$ is a sequence $(f_1, f_2, \cdots, f_k)$ where $f_i \in G_1$ and $t_0(f_i) = s_0(f_{i+1})$:

$$s_0(f_1) \xrightarrow{f_1} t_0(f_1) = s_0(f_2) \xrightarrow{f_2} t_0(f_2) \cdots s_0(f_k) \xrightarrow{f_k} t_0(f_k)$$

For $k = 0$, one gets the empty word $()$.

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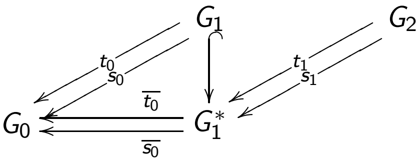
The *free category* over $G$ is the the graph $G^* = (G_0 \underset{\overline{t_0}}{\overset{\overline{s_0}}{\leftleftarrows}} G_1^*)$ where $G_1^*$ is the set of words in $G$ equipped with concatenation as composition, with neutral element given by the empty word:

$$(f_1, f_2, \cdots, f_k)\sharp(g_1, g_2, \cdots, g_r) = (f_1, f_2, \cdots, f_k, g_1, g_2, \cdots, g_r)$$

$$()\sharp(f_1, f_2, \cdots, f_k) = (f_1, f_2, \cdots, f_k) = (f_1, f_2, \cdots, f_k)\sharp()$$

## 2-**polygraphs**

A 2-polygraph is a diagram of sets and functions

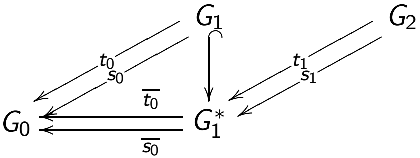



in which $G^* = \left(G_0 \underset{\overline{t_0}}{\overset{\overline{s_0}}{\leftleftarrows}} G_1^*\right)$ is the free category over

$G = \left(G_0 \underset{t_0}{\overset{s_0}{\leftleftarrows}} G_1\right)$ and where $G_0 \underset{\overline{s_0}}{\overset{\overline{t_0}}{\leftleftarrows}} G_1^* \underset{s_1}{\overset{t_1}{\leftleftarrows}} G_2$ is a 2-graph.

## Rewriting systems

A rewriting system is a 2-polygraph




in which the set $G_0$ consists of exactly one element.

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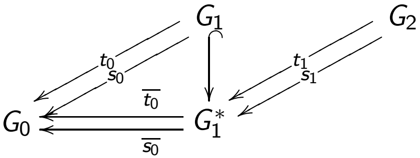


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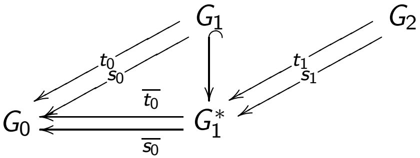


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**Remark:** the emphasis is on the directed nature of $\vDash$.

## The 2-polygraph of $n$-term syllogisms

For every positive natural number $n$:

- $G_0 = \{T_1, \ldots, T_n\}$.
- $G_1 = \{\mathbf{A}_{T_iT_j}, \mathbf{E}_{T_iT_j}, \mathbf{I}_{T_iT_j}, \mathbf{O}_{T_iT_j} \mid 1 \leq i \leq j \leq n\}$
- $G_2$:

| | | |
|---|---|---|
| $(\mathbf{A}_{T_iT_i}) \vDash (\mathbf{A}_{T_iT_i})$ | $(\mathbf{I}_{T_iT_i}) \vDash (\mathbf{I}_{T_iT_i})$ | $1 \leq i \leq n$ |
| | | |
| $(\mathbf{A}_{T_iT_j}) \vDash (\mathbf{A}_{T_iT_j})$ | $(\mathbf{I}_{T_iT_j}) \vDash (\mathbf{I}_{T_iT_j})$ | $1 \leq i < j \leq n$ |
| $(\mathbf{E}_{T_iT_j}) \vDash (\mathbf{E}_{T_iT_j})$ | $(\mathbf{O}_{T_iT_j}) \vDash (\mathbf{O}_{T_iT_j})$ | $1 \leq i < j \leq n$ |
| $(\mathbf{A}_{T_iT_j})\sharp(\mathbf{I}_{T_iT_i}) \vDash \mathbf{I}_{T_iT_j}$ | $(\mathbf{E}_{T_iT_j})\sharp(\mathbf{I}_{T_iT_i}) \vDash \mathbf{O}_{T_iT_j}$ | $1 \leq i < j \leq n$ |
| $(\mathbf{E}_{T_jT_i})^\circ \vDash (\mathbf{E}_{T_iT_j})$ | $(\mathbf{I}_{T_jT_i})^\circ \vDash (\mathbf{I}_{T_iT_j})$ | $1 \leq i < j \leq n$ |
| $(\mathbf{I}_{T_jT_j})\sharp(\mathbf{A}_{T_jT_i})^\circ \vDash (\mathbf{I}_{T_iT_j})$ | $(\mathbf{E}_{T_jT_i})^\circ\sharp(\mathbf{I}_{T_iT_i}) \vDash (\mathbf{O}_{T_iT_j})$ | $1 \leq i < j \leq n$ |
| | | |
| $(\mathbf{A}_{T_jT_k})\sharp(\mathbf{A}_{T_iT_j}) \vDash (\mathbf{A}_{T_iT_k})$ | $(\mathbf{E}_{T_jT_k})\sharp(\mathbf{A}_{T_iT_j}) \vDash (\mathbf{E}_{T_iT_k})$ | $1 \leq i < j < k \leq n$ |
| $(\mathbf{A}_{T_jT_k})\sharp(\mathbf{I}_{T_iT_j}) \vDash (\mathbf{I}_{T_iT_k})$ | $(\mathbf{E}_{T_jT_k})\sharp(\mathbf{I}_{T_iT_j}) \vDash (\mathbf{O}_{T_iT_k})$ | $1 \leq i < j < k \leq n$ |
| $(\mathbf{E}_{T_kT_j})^\circ\sharp(\mathbf{A}_{T_iT_j}) \vDash (\mathbf{E}_{T_iT_k})$ | $(\mathbf{A}_{T_kT_j})^\circ\sharp(\mathbf{E}_{T_iT_j}) \vDash (\mathbf{E}_{T_iT_k})$ | $1 \leq i < j < k \leq n$ |
| $(\mathbf{E}_{T_kT_j})^\circ\sharp(\mathbf{I}_{T_iT_j}) \vDash (\mathbf{O}_{T_iT_k})$ | $(\mathbf{A}_{T_kT_j})^\circ\sharp(\mathbf{O}_{T_iT_j}) \vDash (\mathbf{O}_{T_iT_k})$ | $1 \leq i < j < k \leq n$ |
| $(\mathbf{I}_{T_jT_k})\sharp(\mathbf{A}_{T_jT_i})^\circ \vDash (\mathbf{I}_{T_iT_k})$ | $(\mathbf{A}_{T_jT_k})\sharp(\mathbf{I}_{T_jT_i})^\circ \vDash (\mathbf{I}_{T_iT_k})$ | $1 \leq i < j < k \leq n$ |
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| $(\mathbf{E}_{T_kT_j})^\circ\sharp(\mathbf{I}_{T_jT_i})^\circ \vDash (\mathbf{O}_{T_iT_k})$ | | $1 \leq i < j < k \leq n$ |

**Theorem**

For every positive natural number $n$, the rewriting system for the calculus of $n$-term syllogisms is complete.

## Theorem

For every positive natural number $n$, the rewriting system for the calculus of $n$-term syllogisms is complete.

**Sketch of proof:** It must be observed that the length of the words that undergo rewriting strictly decreases. Then, the proof proceeds by cases depending on $n$.

- $n = 2$: the test is on the two possible rewritings of the word $(\mathbf{E}_{A_jA_i})^\circ \sharp (\mathbf{I}_{A_iA_i})$.
- $n = 3$: the test is on the possible rewritings of the words

$$
\begin{array}{ll}
(\mathbf{E}_{A_kA_j})^\circ \sharp (\mathbf{A}_{A_iA_j}) & (\mathbf{A}_{A_kA_j})^\circ \sharp (\mathbf{E}_{A_jA_i})^\circ \\
(\mathbf{E}_{A_kA_j})^\circ \sharp (\mathbf{I}_{A_iA_j}) & (\mathbf{E}_{A_jA_k}) \sharp (\mathbf{I}_{A_jA_i})^\circ \\
(\mathbf{A}_{A_jA_k}) \sharp (\mathbf{I}_{A_jA_i})^\circ & (\mathbf{E}_{A_jA_k})^\circ \sharp (\mathbf{I}_{A_jA_i})^\circ \\
(\mathbf{I}_{A_kA_j})^\circ \sharp (\mathbf{A}_{A_jA_i})^\circ & (\mathbf{E}_{A_jA_i})^\circ \sharp (\mathbf{I}_{A_iA_i}). \\
(\mathbf{A}_{A_jA_k}) \sharp (\mathbf{A}_{A_iA_j}) \sharp (\mathbf{I}_{A_iA_i}) & (\mathbf{E}_{A_jA_k}) \sharp (\mathbf{A}_{A_iA_j}) \sharp (\mathbf{I}_{A_iA_i}) \\
(\mathbf{A}_{A_kA_j})^\circ \sharp (\mathbf{E}_{A_iA_j}) \sharp (\mathbf{I}_{A_iA_i}) & (\mathbf{E}_{A_kA_j})^\circ \sharp (\mathbf{A}_{A_iA_j}) \sharp (\mathbf{I}_{A_iA_i}) \\
(\mathbf{E}_{A_jA_k}) \sharp (\mathbf{I}_{A_jA_j}) \sharp (\mathbf{A}_{A_jA_i})^\circ & (\mathbf{A}_{A_kA_j})^\circ \sharp (\mathbf{E}_{A_jA_i})^\circ \sharp (\mathbf{I}_{A_iA_i}) \\
(\mathbf{E}_{A_kA_j})^\circ \sharp (\mathbf{I}_{A_jA_j}) \sharp (\mathbf{A}_{A_jA_i})^\circ & (\mathbf{A}_{A_jA_k}) \sharp (\mathbf{I}_{A_jA_j}) \sharp (\mathbf{A}_{A_jA_i})^\circ
\end{array}
$$